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ON SOLVING FUZZY DIFFERENTIAL EQUATIONS BY EXTENDED EULER METHODS BASED ON ROOT MEAN SQUARE AND HARMONIC MEAN

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ABSTRACT

The goal of this research is to provide improved versions of Euler's methods for solving differential equations with fuzzy initial conditions, including Euler's Method, Modified Euler's Method, and Improved Euler's Method with Root Mean Square and Harmonic Mean. To deal with this dependence problem in a fuzzy environment, we update Euler's classical methods using Zadeh's extension concept for fuzzy sets. Finally, we present a numerical example that compares these methods to the traditional Euler's Method and displays that the enhanced methods attain acceptable accuracy.

KEYWORDS: *Fuzzy Initial Value Problem, Euler's Method, Modified Euler's Method, Improved Euler's Method, Root Mean Square, Harmonic Mean.*

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1. INTRODUCTION

By extending the idea of crisp sets to fuzzy sets, fuzzy set theory enables the representation of ambiguity and uncertainty in parameters and boundary conditions, which in turn help us, describe and solve fuzzy differential equations. Because fuzzy differential equations offer a potent tool for simulating real-world systems with inherent ambiguity or incomplete information—which are prevalent in a variety of disciplines, including biology, engineering, and economics—they must be solved. The Euler method, RK method, and other techniques can be used to solve fuzzy differential equations with uncertainty in a computationally efficient manner while maintaining the fuzzy-valued nature of the answers. These days, a lot of scholars are interested in finding solutions to fuzzy differential equations through the extension of traditional numerical methods. Hari Ganesh and Diwahar [**1**] presented a method to solve first-order differential equations with fuzzy initial values by combining the Contra Harmonic Mean with the Centroidal Mean of enhanced Euler's methods. Solving the Heronian and Cubic Mean using an improved version of Euler's Method was suggested by Hari Ganesh and teammates [**2**].Seikkala derivatives based on numerical methods for solving fuzzy ordinary differential equations were introduced by Al Safar and Ibraheem [**3**]. A comprehensive discussion of a numerical approach based on the Runge-Kutta method is followed by an exhaustive error analysis. Fuzzy set-valued functions have the variable order derivative described in the Atangana-Baleanu-Caputo sense. The r – cut representation of a fuzzy-valued function transforms the primary issue under fuzzy beginning conditions into a different problem. In order to address this new issue, we employ operational matrices

(OMs) that are derived from shifted Legendre polynomials (SLPs) developed by Jafari and teammates [**4**]. Mohammed S. Mechee [**5**] has created and adjusted direct explicit numerical RKM techniques for solving classes of higher-order ordinary differential equations (ODEs) to work more efficiently with fuzzy differential equations (FDEs). Khaji and Dawood [**6**] divided the Integro-Differential Equation into two systems of the second order based on the parametric form of the fuzzy number. Shams talked about a brand- new way to solve intuitionistic triangular fuzzy starting point problems in [**7**]. It's called the Generalised Modified Adomian Decomposition Method, and it's used to find the logical answer. Jafaria and teammates [**8**] discussed many numerical approaches for solving fuzzy equations, including dual fuzzy equations, fuzzy differential equations, and fuzzy partial differential equations. Karpagappriya and colleagues [**9**] offered a novel family of cubic spline function methods for solving fuzzy starting value problems effectively. Using the semi-implicit technique to fix the error led to the creation of the better Milstein scheme. The error is the difference between the exact answer of FDE and the result from the Milstein scheme created by Ji and You [**10**]. An andGuo [**11**] discussed the numerical solution to the boundary value problem, which is a two-order fuzzy linear differential equation. Ishak and Chaini [**12**] suggested using the extended trapezoidal method to fix fuzzy difference equations that have first-order initial value problems. Ahmady [**13**] proposed a numerical approach for addressing fuzzy differential equations of fractional order under gH – fractional Caputo differentiability. The method created by Mohammad Alaroud and other researchers [**14**] is based on the generalized Taylor formula and the residual error idea. The Caputo fuzzy H-differentiable is used to characterize the fuzzy fractional derivatives. A numerical method to solve fuzzy differential equations across linear fuzzy real numbers was introduced by Jun [**15**].Kanagarajana [**16**] used the highly generalised differentiability idea to explain a fuzzy differential equation. Ramachandran and Shobanapriya [**17**] compared He's variational iteration technique to the Leapfrog approach for solving first-order linear fuzzy differential equations. Abualhomos [**18**] presented a method for modelling uncertain and incompletely described systems using Runge – Kutta algorithms to solve fuzzy ordinary differential equations. The first-order fuzzy differential equation was developed by Radhy and colleagues [**19**] and was numerically resolved using the Runge – Kutta technique. Jafari and others [**20**] discussed the use of two types of Bernstein neural networks to approximate the solutions of FDEs. Samayan Narayanamoorthy and Yookesh [**21**] proposed an approximate method for solving delay differential equations in the fuzzy domain. Jayakumar and others [**22**] introduced three numerical algorithms to solve fuzzy differential equations. These methods are Milne's explicit five-step technique, implicit four-step, and predictor-corrector, which is the result of combining Milne's explicit five-step methodology and implicit four-step methodologies.

The design of this document is as follows: The basics of fuzzy sets and fuzzy integers are covered in Section 2. We improve on the conventional techniques put out by Euler in Section 3 to get numerical solutions to fuzzy differential equations having less error values. The Root Mean Square and the Harmonic Mean serve as the foundation for these techniques. Section 4 presents the fuzzy adaption of suggested approaches and the fuzzy initial value issue. We demonstrate the effectiveness of our approach in comparison to the conventional Euler's Method in section 5 by using it to provide numerical answers to the fuzzy initial values problem. Section 6 offers a brief conclusion.

2. PRELIMINARIE

To assist readers in understanding the concepts used or defined throughout the study, this section will provide a few explanations of fuzzy sets and fuzzy numbers.

Summary of Definitions

Definition [1]

A fuzzy set $\widetilde{\boldsymbol{A}}$ in the universal set Z is defined as $\widetilde{\boldsymbol{A}} = \{(\boldsymbol{z}, \boldsymbol{\mu}_{\widetilde{A}}(\boldsymbol{z})); \boldsymbol{z} \in \boldsymbol{Z}, \boldsymbol{\mu}_{\widetilde{A}}(\boldsymbol{z}) \in (\boldsymbol{0}, \boldsymbol{1}]\}$. Here, the fuzzy set $\widetilde{\boldsymbol{A}}$'s grade value of $\boldsymbol{z} \in \boldsymbol{Z}$ is $\boldsymbol{\mu}_{\widetilde{A}}(\boldsymbol{z})$, while the membership function's grade is $\boldsymbol{\mu}_{\widetilde{A}}(\boldsymbol{z}) \rightarrow [\boldsymbol{0}, \boldsymbol{1}]$.

Definition [2]

The $\gamma - cut$ of a fuzzy set $\widetilde{\boldsymbol{A}}$ is the crisp set $\tilde{A}^{\gamma}$, which contains all elements of the universal set Z whose membership grades in $\widetilde{\boldsymbol{A}}$ are greater than or equal to the given value of γ. It is shown by

$$\tilde{A}^{\gamma} = \{\boldsymbol{z} \in \boldsymbol{Z} \mid \boldsymbol{\mu}_{\widetilde{A}}(\boldsymbol{z}) \geq \gamma\}$$

Where $0 \leq \gamma \leq 1$.

Definition [3]

The following properties apply to the membership function of a fuzzy number $\widetilde{\boldsymbol{A}}$ that is a subset of real line R:

- $\boldsymbol{\mu}_{\widetilde{A}}(\boldsymbol{z})$ is piecewise continuous in its domain.
- If a $\boldsymbol{z_0} \in \boldsymbol{Z}$ is such that $\boldsymbol{\mu}_{\widetilde{A}}(\boldsymbol{z_0}) = \boldsymbol{1}$, then $\widetilde{\boldsymbol{A}}$ is normal.
- $\widetilde{\boldsymbol{A}}$ is convex, which means that $\boldsymbol{\mu}_{\widetilde{A}}(\boldsymbol{\lambda z_1}(\boldsymbol{1} - \boldsymbol{\lambda})\boldsymbol{z_2}) \geq \boldsymbol{min}(\boldsymbol{\mu}_{\widetilde{A}}(\boldsymbol{z_1}), \boldsymbol{\mu}_{\widetilde{A}}(\boldsymbol{z_2}))and\forall \boldsymbol{z_1}, \boldsymbol{z_2} \in \boldsymbol{Z}$.

Definition [4]

The following is a description of the membership function of a triangular fuzzy number $\widetilde{\boldsymbol{u}}$. The triplet $(s,\ t,\ u)$ may be used to denote it.

$$\widetilde{\boldsymbol{u}}(\boldsymbol{y}) = \begin{cases} \boldsymbol{0}, & if\ \boldsymbol{y} < s \\ \dfrac{\boldsymbol{y} - s}{t - s}, & if\ s\ \leq \boldsymbol{y} \leq t \\ \dfrac{u - \boldsymbol{y}}{u - t}, & if\ t\ \leq \boldsymbol{y} \leq u \\ \boldsymbol{0}, & if\ \boldsymbol{y} > u \end{cases}$$

For each $\gamma \in [0,1]$, the α-level for the fuzzy number $\widetilde{\boldsymbol{u}}$ is $\widetilde{\boldsymbol{u}}_{\gamma} = [s + (t - u)\gamma, u - (u - t)\gamma]$.

3. METHODS FOR EXTENDING EULER'S METHODS UTILIZING THE HARMONIC MEAN AND ROOT MEAN SQUARE

In this phase, we will take over the final steps in solving first-order differential equations with fuzzy initial conditions using the Root mean square and Harmonic mean techniques, together with the Improved Euler and Modified Euler methods.

3.1. Techniques using Extended Euler's Equation Using Root Mean Square

3.1.1 Root Mean SquarebasedEuler's Method

Let us assume that d, e, and f are the three most important variables in a Root Mean sequence. To get the Root Mean Square, use the formula $\boldsymbol{f} = \sqrt{\frac{d^2+e^2}{2}}$. Since these two points are near the centre, we can calculate their predicted Root Mean Square as

$$\left[\left(\frac{{t_0}^2+{t_1}^2}{2}\right)^{1/2},\left(\frac{{v_0}^2+{v_1}^2}{2}\right)^{1/2}\right].$$

Here we define the mathematical equation as

$$\left[\left(\frac{{t_0}^2+(t_0+h)^2}{2}\right)^{1/2},\left(\frac{{v_0}^2+\left(v_0+hf(t_0,v_0)\right)^2}{2}\right)^{1/2}\right].$$

If an equation has a point, like $P(t_0, v_0)$ in the middle of the slope, that needs to be changed, a new equation may be written and given in this way.

$$v_1 = v_0 + hf\left[\left(\frac{{t_0}^2+(t_0+h)^2}{2}\right)^{1/2},\left(\frac{{v_0}^2+\left(v_0+hf(t_0,v_0)\right)^2}{2}\right)^{1/2}\right]$$

The extended Euler's method now yields reliable and accurate findings. The corresponding equation is referred to as the Root Mean Square of Modified Euler's Method. It may be written as:

$$v_{n+1} = v_n + hf\left[\left(\frac{{t_n}^2+(t_n+h)^2}{2}\right)^{1/2},\left(\frac{{v_n}^2+\left(v_n+hf(t_n,v_n)\right)^2}{2}\right)^{1/2}\right]$$

3.1.2 Root Mean Square based Modified Euler's Method

Let us assume that d, e, and f are the three most important variables in a Root Mean sequence. To get the Root Mean Square, use the formula $\boldsymbol{f} = \sqrt{\frac{d^2+e^2}{2}}$. Since these two points are near the centre, we can calculate their predicted RootMean Square as

$$\left[\left(\frac{{t_0}^2+{t_1}^2}{2}\right)^{1/2},\left(\frac{{v_0}^2+{v_1}^2}{2}\right)^{1/2}\right].$$

Here we define the mathematical equation as

$$\left[\left(\frac{{t_0}^2+(t_0+h)^2}{2}\right)^{1/2},\left(\frac{{v_0}^2+\left(v_0+hf\left(t_0+\frac{h}{2},v_0+\frac{h}{2}f(t_0,v_0)\right)\right)^2}{2}\right)^{1/2}\right].$$

If an equation has a point, like $P(t_0, v_0)$ in the middle of the slope, that needs to be changed, a new equation may be written and given in this way.

$$v_1 = v_0 + hf\left[\begin{matrix} t_0+\frac{h}{2}, \\ u_0+\frac{h}{2}f\left[\left(\frac{{t_0}^2+(t_0+h)^2}{2}\right)^{1/2},\left(\frac{{v_0}^2+\left(v_0+hf\left(t_0+\frac{h}{2},v_0+\frac{h}{2}f(t_0,v_0)\right)\right)^2}{2}\right)^{1/2}\right]\end{matrix}\right].$$

The extended Euler's method now yields reliable and accurate findings. The corresponding equation is referred to as the Root Mean Square of Modified Euler's Method. It may be written as:

$$v_{n+1} = v_n + hf\left[\begin{array}{c} t_n + \frac{h}{2}, \\ v_n + \frac{h}{2} f\left[\left(\frac{{t_n}^3+(t_n+h)^2}{2}\right)^{1/2}, \left(\frac{{v_n}^2+\left(v_n+hf\left(t_n+\frac{h}{2}, v_n+\frac{h}{2}f(t_n,v_n)\right)\right)^2}{2}\right)^{1/2}\right]\end{array}\right].$$

3.1.3 Root Mean Square based Improved Euler's Method

Let us assume that d, e, and f are the three most important variables in a Root Mean sequence. To get the Root Mean Square, use the formula $\boldsymbol{f} = \sqrt{\frac{d^2+e^2}{2}}$. Since these two points are near the centre, we can calculate their predicted RootMean Square as

$$\left[\left(\frac{{t_0}^2+{t_1}^2}{2}\right)^{1/2}, \left(\frac{{v_0}^2+{v_1}^2}{2}\right)^{1/2}\right].$$

Here we define the mathematical equation as

$$\left[\left(\frac{{t_0}^2+(t_0+h)^2}{2}\right)^{1/2}, \left(\frac{{v_0}^2+\left(v_0+\frac{h}{2}\left(f(t_0,v_0)+f(t_0+h, v_0+hf(t_0,v_0))\right)\right)^2}{2}\right)^{1/2}\right].$$

If an equation has a point, like $P(t_0, v_0)$ in the middle of the slope, that needs to be changed, a new equation may be written and given in this way.

$$v_1 = v_0 + \frac{h}{2}\left[\begin{array}{c} f\left[\left(\frac{{t_0}^2+(t_0+h)^2}{2}\right)^{1/2}, \left(\frac{{v_0}^2+\left(v_0+\frac{h}{2}\left(f(t_0,v_0)+f(t_0+h, v_0+hf(t_0,v_0))\right)\right)^2}{2}\right)^{1/2}\right] \\ +f\left[\begin{array}{c} t_0 + h, \\ v_n + hf\left[\left(\frac{{t_0}^2+(t_0+h)^2}{2}\right)^{1/2}, \left(\frac{{v_0}^2+\left(v_0+\frac{h}{2}\left(f(t_0,v_0)+f(t_0+h, v_0+hf(t_0,v_0))\right)\right)^2}{2}\right)^{1/2}\right]\end{array}\right]\end{array}\right].$$

The extended Euler's method now yields reliable and accurate findings. The corresponding equation is referred to as the Root Mean Square of Modified Euler's Method. It may be written as:

$$v_{n+1} = v_n + \frac{h}{2}\left[\begin{array}{c} f\left[\left(\frac{{t_n}^2+(t_n+h)^2}{2}\right)^{1/2}, \left(\frac{{v_n}^2+\left(v_n+\frac{h}{2}\left(f(t_n,v_n)+f(t_n+h, v_n+hf(t_n,v_n))\right)\right)^2}{2}\right)^{1/2}\right] \\ +f\left[\begin{array}{c} t_n + h, \\ v_n + hf\left[\left(\frac{{t_n}^2+(t_n+h)^2}{2}\right)^{1/2}, \left(\frac{{v_n}^2+\left(v_n+\frac{h}{2}\left(f(t_n,v_n)+f(t_n+h, v_n+hf(t_n,v_n))\right)\right)^2}{2}\right)^{1/2}\right]\end{array}\right]\end{array}\right].$$

3.2. Techniques using Extended Euler's Equation Using Harmonic Mean:

3.2.1 Harmonic MeanbasedEuler's Method

Consider the following elements of a harmonic sequence: d, e, and f. The harmonic mean can be determined by applying the formula $\boldsymbol{f} = \frac{2de}{d+e}$.Consequently, the anticipated harmonic mean of the two midpoints is expressed as

$$\left[\frac{2t_0t_1}{t_0+t_1}, \frac{2v_0v_1}{v_0+v_1}\right]$$

The equation in maths is defined as

$$\left[\frac{2t_0(t_0+h)}{t_0+(t_0+h)}, \frac{2v_0\big(v_0+hf(t_0,v_0)\big)}{v_0+\big(v_0+hf(t_0,v_0)\big)}\right]$$

If an equation is defined so that it passes through a point, say $P(t_0, v_0)$ in the middle of the slope, we can develop and provide a new equation accordingly.

$$v_1 = v_0 + hf\left[\frac{2t_0(t_0+h)}{t_0+(t_0+h)}, \frac{2v_0\big(v_0+hf(t_0,v_0)\big)}{v_0+\big(v_0+hf(t_0,v_0)\big)}\right]$$

The stability of the Euler method may be improved by estimating at the expected centers of $(t_0, v_0)$and (t_1, v_1) using the modified and stable slope function. The equation described before is known as Euler's Modified Harmonic Mean. It is characterized by

$$v_{n+1} = v_n + hf\left[\frac{2t_n(t_n+h)}{t_n+(t_n+h)}, \frac{2v_n\big(v_n+hf(t_n,v_n)\big)}{v_n+\big(v_n+hf(t_n,v_n)\big)}\right]$$

3.2.2 Harmonic Mean based Modified Euler's Method

Consider the following elements of a harmonic sequence: d, e, and f. The harmonic mean can be determined by applying the formula $\boldsymbol{f} = \frac{2de}{d+e}$.Consequently, the anticipated harmonic mean of the two midpoints is expressed as

$$\left[\frac{2t_0t_1}{t_0+t_1}, \frac{2v_0v_1}{v_0+v_1}\right]$$

The equation in maths is defined as

$$\left[\frac{2t_0(t_0+h)}{t_0+(t_0+h)}, \frac{2v_0\left(v_0+hf\left(t_0+\frac{h}{2}, v_0+\frac{h}{2}f(t_0,v_0)\right)\right)}{v_0+\left(v_0+hf\left(t_0+\frac{h}{2}, v_0+\frac{h}{2}f(t_0,v_0)\right)\right)}\right]$$

If an equation is defined so that it passes through a point, say $P(t_0, v_0)$ in the middle of the slope, we can develop and provide a new equation accordingly.

$$v_1 = v_0 + hf\left[\begin{matrix} t_0+\frac{h}{2}, \\ v_0+\frac{h}{2}f\left[\frac{2t_0(t_0+h)}{t_0+(t_0+h)}, \frac{2v_0\left(v_0+hf\left(t_0+\frac{h}{2}, v_0+\frac{h}{2}f(t_0,v_0)\right)\right)}{v_0+\left(v_0+hf\left(t_0+\frac{h}{2}, v_0+\frac{h}{2}f(t_0,v_0)\right)\right)}\right]\end{matrix}\right].$$

The stability of the Euler method may be improved by estimating at the expected centers of $(t_0, v_0)$and (t_1, v_1) using the modified and stable slope function. The equation described before is known as Euler's Modified Harmonic Mean. It is characterized by

$$v_{n+1} = v_n + hf\left[\begin{matrix} t_n + \frac{h}{2}, \\ v_n + \frac{h}{2} f\left[\frac{2t_n(t_n+h)}{t_n+(t_n+h)}, \frac{2v_n\left(v_n+hf\left(t_n+\frac{h}{2}, v_n+\frac{h}{2}f(t_n,v_n)\right)\right)}{v_n+\left(v_n+hf\left(t_n+\frac{h}{2}, v_n+\frac{h}{2}f(t_n,v_n)\right)\right)}\right]\end{matrix}\right].$$

3.2.3 Harmonic Mean based Improved Euler's Method

Consider the following elements of a harmonic sequence: d, e, and f. The harmonic mean can be determined by applying the formula $\boldsymbol{f} = \frac{2de}{d+e}$.Consequently, the anticipated harmonic mean of the two midpoints is expressed as

$$\left[\frac{2t_0t_1}{t_0 + t_1}, \frac{2v_0v_1}{v_0 + v_1}\right]$$

The equation in maths is defined as

$$\left[\frac{2t_0(t_0 + h)}{t_0 + (t_0 + h)}, \frac{2v_0\left(v_0 + \frac{h}{2}\left(f(t_0, v_0) + f\left(t_0 + h, v_0 + hf(t_0, v_0)\right)\right)\right)}{v_0 + \left(v_0 + \frac{h}{2}\left(f(t_0, v_0) + f\left(t_0 + h, v_0 + hf(t_0, v_0)\right)\right)\right)}\right]$$

If an equation is defined so that it passes through a point, say $P(t_0, v_0)$ in the middle of the slope, we can develop and provide a new equation accordingly.

$$v_1 = v_0 + \frac{h}{2}\left[\begin{matrix} f\left[\frac{2t_0(t_0+h)}{t_0+(t_0+h)}, \frac{2v_0\left(v_0+\frac{h}{2}\left(f(t_0,v_0)+f\left(t_0+h,v_0+hf(t_0,v_0)\right)\right)\right)}{v_0+\left(v_0+\frac{h}{2}\left(f(t_0,v_0)+f\left(t_0+h,v_0+hf(t_0,v_0)\right)\right)\right)}\right] \\ +f\left[\begin{matrix} t_0 + h, \\ v_0 + hf\left[\frac{2t_0(t_0+h)}{t_0+(t_0+h)}, \frac{2v_0\left(v_0+\frac{h}{2}\left(f(t_0,v_0)+f\left(t_0+h,v_0+hf(t_0,v_0)\right)\right)\right)}{v_0+\left(v_0+\frac{h}{2}\left(f(t_0,v_0)+f\left(t_0+h,v_0+hf(t_0,v_0)\right)\right)\right)}\right]\end{matrix}\right]\end{matrix}\right].$$

The stability of the Euler method may be improved by estimating at the expected centers of $(t_0, v_0)$and (t_1, v_1) using the modified and stable slope function. The equation described before is known as Euler's Modified Harmonic Mean. It is characterized by

$$v_{n+1} = v_n + \frac{h}{2}\left[\begin{matrix} f\left[\frac{2t_n(t_n+h)}{t_n+(t_n+h)}, \frac{2v_n\left(v_n+\frac{h}{2}\left(f(t_n,v_n)+f\left(t_n+h,v_n+hf(t_n,v_n)\right)\right)\right)}{v_n+\left(v_n+\frac{h}{2}\left(f(t_n,v_n)+f\left(t_n+h,v_n+hf(t_n,v_n)\right)\right)\right)}\right] \\ +f\left[\begin{matrix} t_n + h, \\ v_n + hf\left(\frac{2t_n(t_n+h)}{t_n+(t_n+h)}, \frac{2v_n\left(v_n+\frac{h}{2}\left(f(t_n,v_n)+f\left(t_n+h,v_n+hf(t_n,v_n)\right)\right)\right)}{v_n+\left(v_n+\frac{h}{2}\left(f(t_n,v_n)+f\left(t_n+h,v_n+hf(t_n,v_n)\right)\right)\right)}\right)\end{matrix}\right]\end{matrix}\right].$$

4. A PROBLEM WITH FUZZY INITIAL VALUES

4.1 Method for Solving FIVP with High Accuracy

Check out the FIVP

$$v'(t) = \begin{cases} f(t,v) \\ v(t_0) = (\underline{v}_0, v_0^m, \overline{v}_0) \end{cases}$$

Here, fuzzy triangles are used to illustrate a fuzzy initial condition. Using fuzzy initial conditions, the FIVP obtains its solution.

4.2 Strategies for FIVP

In this study, we examine the fuzzy initial value problem

$$v'(t) = \begin{cases} f(t,v) \\ v(t_0) = (\underline{v}_0, v_0^m, \overline{v}_0) \end{cases}$$

The triangular fuzzy initial condition may be expressed using the s-cut approach as

$$[(v^m - \underline{v}_0)\gamma + \underline{v}_0, \overline{v}_0 + (v^m - \overline{v}_0)\gamma], 0 \leq \gamma \leq 1.$$

An improved version of Euler's methods was supposed to solve the FIVP. These days, we compute every potential new upper and lower bounds using the Extended Euler's methods. Below you can see the output of the grid point calculations with respect to t.

$$\underline{v}_{n+1}^{(1)}(t_{n+1}:\gamma) = \underline{v}(t_n:\gamma) + F[v(t_n:\gamma)],$$

$$\overline{v}_{n+1}^{(1)}(t_{n+1}:\gamma) = \overline{v}(t_n:\gamma) + G[v(t_n:\gamma)],$$

$$\underline{v}_{n+1}^{(2)}(t_{n+1}:\gamma) = \underline{v}(t_n:\gamma) + G[v(t_n:\gamma)],$$

$$\overline{v}_{n+1}^{(2)}(t_{n+1}:\gamma) = \overline{v}(t_n:\gamma) + F[v(t_n:\gamma)],$$

Next, we sort the values from lowest to highest, which represent the lower and upper bounds of the variable v, respectively, so that it can provide accuracy and better results.

$$\underline{v}_{n+1} = min\{\underline{v}(t_n:\gamma) + F[v(t_n:\gamma)], \underline{v}(t_n:\gamma) + G[v(t_n:\gamma)]\},$$

$$\overline{v}_{n+1} = max\{\overline{v}(t_n:\gamma) + G[v(t_n:\gamma)], \overline{v}(t_n:\gamma) + F[v(t_n:\gamma)]\}.$$

4.2.1. Extended Euler's Methods for FIVP utilizing Root Mean Square

4.2.1.1 Root Mean Square-based Euler's Method for FIVP

The suggested Root Mean Square for the Extended Euler's technique is exactly specified by

$$\underline{v}_{n+1} = \underline{v}(t_n:\gamma) + F(v(t_n:\gamma)),$$

$$\overline{v}_{n+1} = \overline{v}(t_n:\gamma) + G(v(t_n:\gamma)).$$

Where

$$F = hf\left[\left(\frac{(t_n:\gamma)^2+((t_n:\gamma)+h)^2}{2}\right)^{1/2}, \left(\frac{(\underline{v}_n:\gamma)^2+\left((\underline{v}_n:\gamma)+hf\left((t_n:\gamma),(\underline{v}_n:\gamma)\right)\right)^2}{2}\right)^{1/2}\right],$$

$$G = hf\left[\left(\frac{(t_n:\gamma)^2+((t_n:\gamma)+h)^2}{2}\right)^{1/2}, \left(\frac{(\overline{v}_n:\gamma)^2+\left((\overline{v}_n:\gamma)+hf\left((t_n:\gamma),(\overline{v}_n:\gamma)\right)\right)^2}{2}\right)^{1/2}\right].$$

4.2.1.2 Root Mean Square-basedModified Euler's Method for FIVP

The suggested Root Mean Square for the Extended Modified Euler's technique is exactly specified by

$$\underline{v}_{n+1} = \underline{v}(t_n:\gamma) + F\big(v(t_n:\gamma)\big),$$

$$\overline{v}_{n+1} = \overline{v}(t_n:\gamma) + G\big(v(t_n:\gamma)\big).$$

Where

$$F = hf\left[\begin{matrix} (t_n:\gamma)+\frac{h}{2}, \\ (\underline{v}_n:\gamma)+\frac{h}{2}f\left[\left(\frac{(t_n:\gamma)^2+((t_n:\gamma)+h)^2}{2}\right)^{1/2}, \left(\frac{(\underline{v}_n:\gamma)^2+\left((\underline{v}_n:\gamma)+hf\left((t_n:\gamma)+\frac{h}{2},(\underline{v}_n:\gamma)+\frac{h}{2}f\left((t_n:\gamma),(\underline{v}_n:\gamma)\right)\right)\right)^2}{2}\right)^{1/2}\right]\end{matrix}\right],$$

$$G = hf\left[\begin{matrix} (t_n:\gamma)+\frac{h}{2}, \\ (\overline{v}_n:\gamma)+\frac{h}{2}f\left[\left(\frac{(t_n:\gamma)^2+((t_n:\gamma)+h)^2}{2}\right)^{1/2}, \left(\frac{(\overline{v}_n:\gamma)^2+\left((\overline{v}_n:\gamma)+hf\left((t_n:\gamma)+\frac{h}{2},(\overline{v}_n:\gamma)+\frac{h}{2}f\left((t_n:\gamma),(\overline{v}_n:\gamma)\right)\right)\right)^2}{2}\right)^{1/2}\right]\end{matrix}\right].$$

4.2.1.3 Root Mean Square-basedImproved Euler's Method for FIVP

The suggested Root Mean Square for the Extended ImprovedEuler's technique is exactly specified by

$$\underline{v}_{n+1} = \underline{v}(t_n:\gamma) + F\big(v(t_n:\gamma)\big),$$

$$\overline{v}_{n+1} = \overline{v}(t_n:\gamma) + G\big(v(t_n:\gamma)\big).$$

Where

$$F=\frac{h}{2}\left[\begin{array}{c} f\left[\begin{array}{c}\left(\frac{(t_n:\gamma)^2+((t_n:\gamma)+h)^2}{2}\right)^{1/2},\\ \left(\frac{(\underline{v}_n:\gamma)^2+\left((\underline{v}_n:\gamma)+\frac{h}{2}\left(f((t_n:\gamma),(\underline{v}_n:\gamma))+f\left((t_n:\gamma)+h,(\underline{v}_n:\gamma)+hf\left((t_n:\gamma),(\underline{v}_n:\gamma)\right)\right)\right)\right)^2}{2}\right)^{1/2}\end{array}\right]\\ +f\left(\begin{array}{c}(t_n:\gamma)+h,\\ (\underline{v}_n:\gamma)+hf\left[\begin{array}{c}\left(\frac{(t_n:\gamma)^2+((t_n:\gamma)+h)^2}{2}\right)^{1/2},\\ \left(\frac{(\underline{v}_n:\gamma)^2+\left((\underline{v}_n:\gamma)+\frac{h}{2}\left(f((t_n:\gamma),(\underline{v}_n:\gamma))+f\left((t_n:\gamma)+h,(\underline{v}_n:\gamma)+hf\left((t_n:\gamma),(\underline{v}_n:\gamma)\right)\right)\right)\right)^2}{2}\right)^{1/2}\end{array}\right]\end{array}\right)\end{array}\right],$$

$$G=\frac{h}{2}\left[\begin{array}{c} f\left[\begin{array}{c}\left(\frac{(t_n:\gamma)^2+((t_n:\gamma)+h)^2}{2}\right)^{1/2},\\ \left(\frac{(\overline{v}_n:\gamma)^2+\left((\overline{v}_n:\gamma)+\frac{h}{2}\left(f((t_n:\gamma),(\overline{v}_n:\gamma))+f\left((t_n:\gamma)+h,(\overline{v}_n:\gamma)+hf\left((t_n:\gamma),(\overline{v}_n:\gamma)\right)\right)\right)\right)^2}{2}\right)^{1/2}\end{array}\right]\\ +f\left(\begin{array}{c}(t_n:\gamma)+h,\\ (\overline{v}_n:\gamma)+hf\left[\begin{array}{c}\left(\frac{(t_n:\gamma)^2+((t_n:\gamma)+h)^2}{2}\right)^{1/2},\\ \left(\frac{(\overline{v}_n:\gamma)^2+\left((\overline{v}_n:\gamma)+\frac{h}{2}\left(f((t_n:\gamma),(\overline{v}_n:\gamma))+f\left((t_n:\gamma)+h,(\overline{v}_n:\gamma)+hf\left((t_n:\gamma),(\overline{v}_n:\gamma)\right)\right)\right)\right)^2}{2}\right)^{1/2}\end{array}\right]\end{array}\right)\end{array}\right].$$

4.2.2. Extended Euler's Methods for FIVP utilizing Harmonic Mean

4.2.2.1Harmonic Mean basedEuler's Method for FIVP

The exact definition of the imagined Harmonic Mean for the Extended Euler's technique is

$$\underline{v}_{n+1}=\underline{v}(t_n:\gamma)+F\big(v(t_n:\gamma)\big),$$

$$\overline{v}_{n+1}=\overline{v}(t_n:\gamma)+G\big(v(t_n:\gamma)\big).$$

Where

$$F=hf\left(\frac{2(t_n:\gamma)((t_n:\gamma)+h)}{(t_n:\gamma)+((t_n:\gamma)+h)},\frac{2(\underline{v}_n:\gamma)\left((\underline{v}_n:\gamma)+hf\left((t_n:\gamma),(\underline{v}_n:\gamma)\right)\right)}{(\underline{v}_n:\gamma)+\left((\underline{v}_n:\gamma)+hf\left((t_n:\gamma),(\underline{v}_n:\gamma)\right)\right)}\right),$$

$$G=hf\left(\frac{2(t_n:\gamma)((t_n:\gamma)+h)}{(t_n:\gamma)+((t_n:\gamma)+h)},\frac{2(\overline{v}_n:\gamma)\left((\overline{v}_n:\gamma)+hf\left((t_n:\gamma),(\overline{v}_n:\gamma)\right)\right)}{(\overline{v}_n:\gamma)+\left((\overline{v}_n:\gamma)+hf\left((t_n:\gamma),(\overline{v}_n:\gamma)\right)\right)}\right).$$

4.2.2.2 Harmonic Mean based Modified Euler's Method for FIVP

The exact definition of the imagined Harmonic Mean for the Extended Modified Euler's technique is

$$\underline{v}_{n+1}=\underline{v}(t_n:\gamma)+F\big(v(t_n:\gamma)\big),$$

$$\overline{v}_{n+1} = \overline{v}(t_n{:}\gamma) + G\big(v(t_n{:}\gamma)\big).$$

Where

$$F = hf\left(\begin{array}{c} (t_n{:}\gamma) + \frac{h}{2}, \\ (\underline{v}_n{:}\gamma) + \frac{h}{2} f\left(\frac{2(t_n{:}\gamma)((t_n{:}\gamma)+h)}{(t_n{:}\gamma)+((t_n{:}\gamma)+h)}, \frac{2(\underline{v}_n{:}\gamma)\left((\underline{v}_n{:}\gamma)+hf\left((t_n{:}\gamma)+\frac{h}{2},(\underline{v}_n{:}\gamma)+\frac{h}{2}f\left((t_n{:}\gamma),(\underline{v}_n{:}\gamma)\right)\right)\right)}{(\underline{v}_n{:}\gamma)+\left((\underline{v}_n{:}\gamma)+hf\left((t_n{:}\gamma)+\frac{h}{2},(\underline{v}_n{:}\gamma)+\frac{h}{2}f\left((t_n{:}\gamma),(\underline{v}_n{:}\gamma)\right)\right)\right)} \right) \end{array} \right),$$

$$G = hf\left(\begin{array}{c} (t_n{:}\gamma) + \frac{h}{2}, \\ (\overline{v}_n{:}\gamma) + \frac{h}{2} f\left(\frac{2(t_n{:}\gamma)((t_n{:}\gamma)+h)}{(t_n{:}\gamma)+((t_n{:}\gamma)+h)}, \frac{2(\overline{v}_n{:}\gamma)\left((\overline{v}_n{:}\gamma)+hf\left((t_n{:}\gamma)+\frac{h}{2},(\overline{v}_n{:}\gamma)+\frac{h}{2}f\left((t_n{:}\gamma),(\overline{v}_n{:}\gamma)\right)\right)\right)}{(\overline{v}_n{:}\gamma)+\left((\overline{v}_n{:}\gamma)+hf\left((t_n{:}\gamma)+\frac{h}{2},(\overline{v}_n{:}\gamma)+\frac{h}{2}f\left((t_n{:}\gamma),(\overline{v}_n{:}\gamma)\right)\right)\right)} \right) \end{array} \right).$$

4.2.2.3 Harmonic Mean based Improved Euler's Method for FIVP

The exact definition of the imagined Harmonic Mean for the Extended Improved Euler's technique is

$$\underline{v}_{n+1} = \underline{v}(t_n{:}\gamma) + F\big(v(t_n{:}\gamma)\big),$$

$$\overline{v}_{n+1} = \overline{v}(t_n{:}\gamma) + G\big(v(t_n{:}\gamma)\big).$$

Where

$$F = \frac{h}{2}\left[\begin{array}{c} f\left[\frac{2(t_n{:}\gamma)((t_n{:}\gamma)+h)}{(t_n{:}\gamma)+((t_n{:}\gamma)+h)}, \frac{2(\underline{v}_n{:}\gamma)\left((\underline{v}_n{:}\gamma)+\frac{h}{2}\left(f\left((t_n{:}\gamma),(\underline{v}_n{:}\gamma)\right)+f\left((t_n{:}\gamma)+h,(\underline{v}_n{:}\gamma)+hf\left((t_n{:}\gamma),(\underline{v}_n{:}\gamma)\right)\right)\right)\right)}{(\underline{v}_n{:}\gamma)+\left((\underline{v}_n{:}\gamma)+\frac{h}{2}\left(f\left((t_n{:}\gamma),(\underline{v}_n{:}\gamma)\right)+f\left((t_n{:}\gamma)+h,(\underline{v}_n{:}\gamma)+hf\left((t_n{:}\gamma),(\underline{v}_n{:}\gamma)\right)\right)\right)\right)} \right] \\ +f\left(\begin{array}{c} (t_n{:}\gamma) + h, \\ (\underline{v}_n{:}\gamma) + hf\left[\frac{2(t_n{:}\gamma)((t_n{:}\gamma)+h)}{(t_n{:}\gamma)+((t_n{:}\gamma)+h)}, \frac{2(\underline{v}_n{:}\gamma)\left((\underline{v}_n{:}\gamma)+\frac{h}{2}\left(f\left((t_n{:}\gamma),(\underline{v}_n{:}\gamma)\right)+f\left((t_n{:}\gamma)+h,(\underline{v}_n{:}\gamma)+hf\left((t_n{:}\gamma),(\underline{v}_n{:}\gamma)\right)\right)\right)\right)}{(\underline{v}_n{:}\gamma)+\left((\underline{v}_n{:}\gamma)+\frac{h}{2}\left(f\left((t_n{:}\gamma),(\underline{v}_n{:}\gamma)\right)+f\left((t_n{:}\gamma)+h,(\underline{v}_n{:}\gamma)+hf\left((t_n{:}\gamma),(\underline{v}_n{:}\gamma)\right)\right)\right)\right)} \right] \end{array} \right) \end{array} \right],$$

$$G = \frac{h}{2}\left[\begin{array}{c} f\left[\frac{2(t_n{:}\gamma)((t_n{:}\gamma)+h)}{(t_n{:}\gamma)+((t_n{:}\gamma)+h)}, \frac{2(\overline{v}_n{:}\gamma)\left((\overline{v}_n{:}\gamma)+\frac{h}{2}\left(f\left((t_n{:}\gamma),(\overline{v}_n{:}\gamma)\right)+f\left((t_n{:}\gamma)+h,(\overline{v}_n{:}\gamma)+hf\left((t_n{:}\gamma),(\overline{v}_n{:}\gamma)\right)\right)\right)\right)}{(\overline{v}_n{:}\gamma)+\left((\overline{v}_n{:}\gamma)+\frac{h}{2}\left(f\left((t_n{:}\gamma),(\overline{v}_n{:}\gamma)\right)+f\left((t_n{:}\gamma)+h,(\overline{v}_n{:}\gamma)+hf\left((t_n{:}\gamma),(\overline{v}_n{:}\gamma)\right)\right)\right)\right)} \right] \\ +f\left(\begin{array}{c} (t_n{:}\gamma) + h, \\ (\overline{v}_n{:}\gamma) + hf\left[\frac{2(t_n{:}\gamma)((t_n{:}\gamma)+h)}{(t_n{:}\gamma)+((t_n{:}\gamma)+h)}, \frac{2(\overline{v}_n{:}\gamma)\left((\overline{v}_n{:}\gamma)+\frac{h}{2}\left(f\left((t_n{:}\gamma),(\overline{v}_n{:}\gamma)\right)+f\left((t_n{:}\gamma)+h,(\overline{v}_n{:}\gamma)+hf\left((t_n{:}\gamma),(\overline{v}_n{:}\gamma)\right)\right)\right)\right)}{(\overline{v}_n{:}\gamma)+\left((\overline{v}_n{:}\gamma)+\frac{h}{2}\left(f\left((t_n{:}\gamma),(\overline{v}_n{:}\gamma)\right)+f\left((t_n{:}\gamma)+h,(\overline{v}_n{:}\gamma)+hf\left((t_n{:}\gamma),(\overline{v}_n{:}\gamma)\right)\right)\right)\right)} \right] \end{array} \right) \end{array} \right].$$

5. EXAMPLE IN NUMERICAL FORM

Take the differential equation with the fuzzy initial value problem as an example.

$$v'(t) = t^3(v),$$

$$v(0) = (0.96 + 0.04\gamma, 1.01 - 0.01\gamma)\ Where\ 0 \leq \gamma \leq 1.$$

Solution

The accurate answer is given by

$$\underline{v}(t:\gamma) = \underline{v}(t:\gamma)e^{\frac{t^4}{4}} \text{and } \overline{v}(t:\gamma) = \overline{v}(t:\gamma)e^{\frac{t^4}{4}},$$

Following that, the answers given by $t = 1$,

$$\overline{v}(1:\gamma) = [(0.96 + 0.04\gamma)e^{0.25}, (1.01 - 0.01\gamma)e^{0.25}], 0 \le \gamma \le 1.$$

(A) Extended Euler's Techniques: Combining Root Mean Square and Harmonic Mean

Euler's method generates a solution that is both exact and somewhat accurate. The Root Mean Square and Harmonic Mean of the Euler, Modified Euler, and Improved Euler's Methods utilizing $h = 0.1$ and $h = 0.001$ are shown below:

(A1) Euler's Method in Combination with Root Mean Square and Harmonic Mean

For various values of $\gamma \in [0,1]$, the approximate solutions derived using Root Mean Square and Harmonic Mean of Euler's method for $h = 0.1$ and $h = 0.001$ are shown below:

Table 1.1: Numerical Solutions for Euler's Method using Root Mean Square and Harmonic Mean with $h = 0.1$

γ	T	Exact Solutions		Euler's Method		Euler's Method using Root Mean Square		Euler's Method using Harmonic Mean	
		LOWER	UPPER	Lower	Upper	Lower	Upper	Lower	Upper
0	1	1.2326644000	1.2968656709	1.1699589886	1.2308943526	1.2319597032	1.2961242711	1.2249959201	1.2887977909
0.1	1	1.2378005017	1.2955816454	1.1748338177	1.2296756453	1.2370928687	1.2948409798	1.2301000697	1.2875217535
0.2	1	1.2429366034	1.2942976200	1.1797086468	1.2284569380	1.2422260341	1.2935576884	1.2352042194	1.2862457161
0.3	1	1.2480727050	1.2930135946	1.1845834759	1.2272382307	1.2473591995	1.2922743970	1.2403083691	1.2849696786
0.4	1	1.2532088067	1.2917295692	1.1894583051	1.2260195235	1.2524923650	1.2909911057	1.2454125187	1.2836936412
0.5	1	1.2583449084	1.2904455438	1.1943331342	1.2248008162	1.2576255304	1.2897078143	1.2505166684	1.2824176038
0.6	1	1.2634810100	1.2891615184	1.1992079633	1.2235821089	1.2627586958	1.2884245230	1.2556208181	1.2811415664
0.7	1	1.2686171117	1.2878774929	1.2040827924	1.2223634016	1.2678918612	1.2871412316	1.2607249677	1.2798655290
0.8	1	1.2737532134	1.2865934675	1.2089576215	1.2211446943	1.2730250267	1.2858579403	1.2658291174	1.2785894916
0.9	1	1.2788893150	1.2853094421	1.2138324507	1.2199259871	1.2781581921	1.2845746489	1.2709332671	1.2773134541
1	1	1.2840254167	1.2840254167	1.2187072798	1.2187072798	1.2832913575	1.2832913575	1.2760374167	1.2760374167

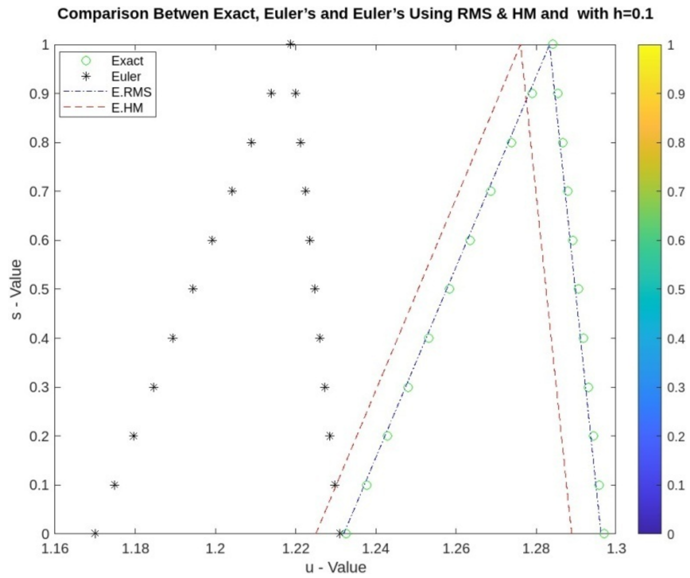

Figure 1.1: Graphical Representation of Solutions for Euler's Method using Root Mean Square and Harmonic Mean with $h = 0.1$

Table 1.2: Absolute Error Analysis for Euler's Method using Root Mean Square and Harmonic Mean with $h = 0.1$

γ	Euler's Method		Euler's Method using Root Mean Square		Euler's Method using Harmonic Mean	
	LOWER	UPPER	Lower	Upper	Lower	Upper
0	0.0627054114	0.0659713183	0.0007046968	0.0007413998	0.0076684799	0.0080678800
0.1	0.0629666840	0.0659060001	0.0007076330	0.0007406656	0.0077004320	0.0080598919
0.2	0.0632279566	0.0658406820	0.0007105693	0.0007399316	0.0077323840	0.0080519039
0.3	0.0634892291	0.0657753639	0.0007135055	0.0007391976	0.0077643359	0.0080439160
0.4	0.0637505016	0.0657100457	0.0007164417	0.0007384635	0.0077962880	0.0080359280
0.5	0.0640117742	0.0656447276	0.0007193780	0.0007377295	0.0078282400	0.0080279400
0.6	0.0642730467	0.0655794095	0.0007223142	0.0007369954	0.0078601919	0.0080199520
0.7	0.0645343193	0.0655140913	0.0007252505	0.0007362613	0.0078921440	0.0080119639
0.8	0.0647955919	0.0654487732	0.0007281867	0.0007355272	0.0079240960	0.0080039759
0.9	0.0650568643	0.0653834550	0.0007311229	0.0007347932	0.0079560479	0.0079959880
1	0.0653181369	0.0653181369	0.0007340592	0.0007340592	0.0079880000	0.0079880000

Table 1.3: Percentage Relative Error Analysis for Euler's Method using Root Mean Square and Harmonic Mean with h = 0.1

γ	Euler's Method	Euler's Method using Root Mean Square	Euler's Method using Harmonic Mean
	(Lower, Upper)	**(Lower, Upper)**	**(Lower, Upper)**
$\gamma \in [0, 1]$	5.09%	0.06%	0.62%

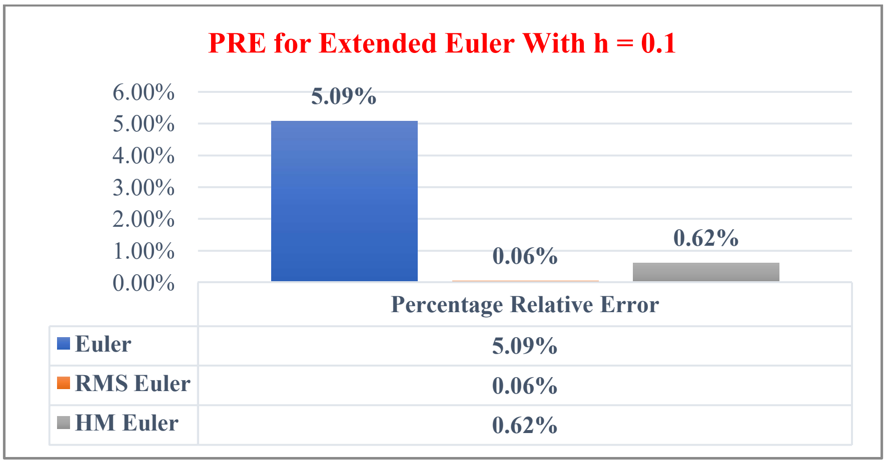

Figure 1.3: Graphical Representation of Percentage Relative Error Analysisfor Euler's Method using Root Mean Squareand Harmonic Mean with h = 0.1

Table 2.1: Numerical Solutions for Euler's Method using Root Mean Square and Harmonic Mean with h = 0.001

γ	T	Exact Solutions		Euler's Method		Euler's Method using Root Mean Square		Euler's Method using Harmonic Mean	
		LOWER	**UPPER**	**Lower**	**Upper**	**Lower**	**Upper**	**Lower**	**Upper**
0	**1**	1.2326644000	1.2968656709	0.9619454992	1.0120468273	0.9619500859	1.0120516529	0.9619500809	1.0120516476
0.1	**1**	1.2378005017	1.2955816454	0.9659536055	1.0110448008	0.9659582113	1.0110496216	0.9659582062	1.0110496162
0.2	**1**	1.2429366034	1.2942976200	0.9699617117	1.0100427742	0.9699663366	1.0100475902	0.9699663315	1.0100475849
0.3	**1**	1.2480727050	1.2930135946	0.9739698180	1.0090407476	0.9739744620	1.0090455589	0.9739744569	1.0090455536
0.4	**1**	1.2532088067	1.2917295692	0.9779779242	1.0080387211	0.9779825874	1.0080435276	0.9779825822	1.0080435222
0.5	**1**	1.2583449084	1.2904455438	0.9819860305	1.0070366945	0.9819907127	1.0070414962	0.9819907075	1.0070414909
0.6	**1**	1.2634810100	1.2891615184	0.9859941367	1.0060346679	0.9859988381	1.0060394649	0.9859988329	1.0060394596
0.7	**1**	1.2686171117	1.2878774929	0.9900022430	1.0050326414	0.9900069634	1.0050374335	0.9900069582	1.0050374282
0.8	**1**	1.2737532134	1.2865934675	0.9940103492	1.0040306148	0.9940150888	1.0040354022	0.9940150835	1.0040353969
0.9	**1**	1.2788893150	1.2853094421	0.9980184554	1.0030285883	0.9980232142	1.0030333709	0.9980232089	1.0030333656
1	**1**	1.2840254167	1.2840254167	1.0020265617	1.0020265617	1.0020313395	1.0020313395	1.0020313342	1.0020313342

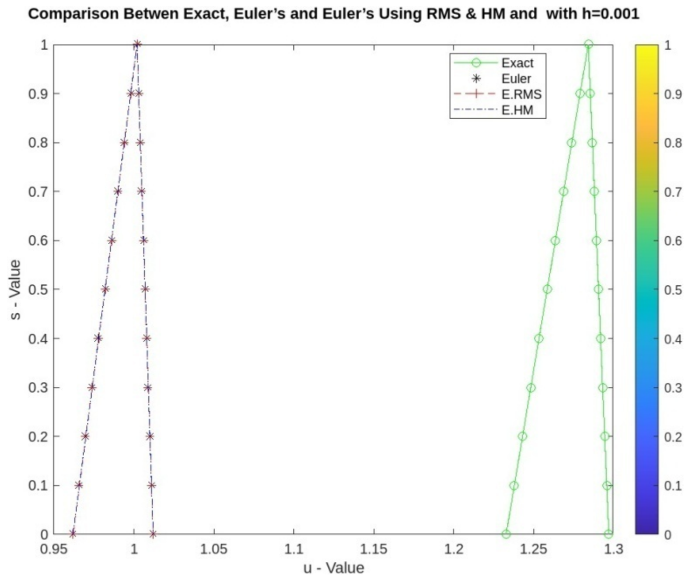

Figure 2.1: Graphical Representation of Solutions for Euler's Method using Root Mean Square and Harmonic Mean with $h = 0.001$

Table 2.2: Error Analysis for Euler's Method using Root Mean Square and Harmonic Mean with h = 0.001

γ	Euler's Method		Euler's Method using Root Mean Square		Euler's Method using Harmonic Mean	
	LOWER	**UPPER**	**Lower**	**Upper**	**Lower**	**Upper**
0	0.2707189008	0.2848188436	0.2707143141	0.2848140180	0.2707143191	0.2848140233
0.1	0.2718468962	0.2845368446	0.2718422904	0.2845320238	0.2718422955	0.2845320292
0.2	0.2729748917	0.2842548458	0.2729702668	0.2842500298	0.2729702719	0.2842500351
0.3	0.2741028870	0.2839728470	0.2740982430	0.2839680357	0.2740982481	0.2839680410
0.4	0.2752308825	0.2836908481	0.2752262193	0.2836860416	0.2752262245	0.2836860470
0.5	0.2763588779	0.2834088493	0.2763541957	0.2834040476	0.2763542009	0.2834040529
0.6	0.2774868733	0.2831268505	0.2774821719	0.2831220535	0.2774821771	0.2831220588
0.7	0.2786148687	0.2828448515	0.2786101483	0.2828400594	0.2786101535	0.2828400647
0.8	0.2797428642	0.2825628527	0.2797381246	0.2825580653	0.2797381299	0.2825580706
0.9	0.2808708596	0.2822808538	0.2808661008	0.2822760712	0.2808661061	0.2822760765
1	0.2819988550	0.2819988550	0.2819940772	0.2819940772	0.2819940825	0.2819940825

Table 2.3: Percentage Relative Error Analysis for Euler's Method using Root Mean Square and Harmonic Mean with h = 0.001

γ	**Euler's Method**	**Euler's Method using Root Mean Square**	**Euler's Method using Harmonic Mean**
	(Lower, Upper)	**(Lower, Upper)**	**(Lower, Upper)**
$\gamma \in [0, 1]$	21.96%	21.96%	21.96%

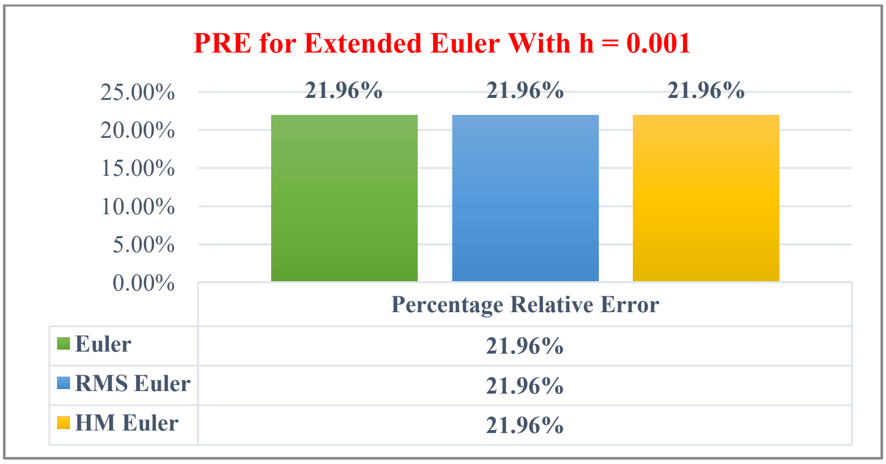

Figure 2.3: Graphical Representation of Percentage Relative Error Analysisfor Euler's Method using Root Mean Squareand Harmonic Mean with h = 0.001.

Tables 1.3, 2.3, and figures 1.3, 2.3 indicate that the findings obtained via the Extended Euler Method utilizing the Root Mean Square and Harmonic Mean are closely aligned with the precise value. This methodology may provide more accurate findings than the conventional Euler method. The case study for the two scenarios h = 0.1 and h = 0.001 reveals that the suggested Euler methods minimize error levels well. Consequently, the approaches are superior to the conventional approach.

(A2) *Modified Euler's Method in Combination with Root Mean Square and Harmonic Mean:*

For various values of$\gamma \in [0,1]$, the approximate solutions derived using Root Mean Square and Harmonic Mean of Modified Euler's method for $h = 0.1$ and $h = 0.001$ are shown below:

Table 3.1: Numerical Solutions for ModifiedEuler's Method using Root Mean Square and Harmonic Mean with $h = 0.1$

γ	T	Exact Solutions		Euler's Method		Modified Euler's Method usingRoot Mean Square		Modified Euler's Method usingHarmonic Mean	
		LOWER	UPPER	Lower	Upper	Lower	Upper	Lower	Upper
0	1	1.2326644000	1.2968656709	1.1699589886	1.2308943526	1.2312711661	1.2953998727	1.2311252311	1.2952463369
0.1	1	1.2378005017	1.2955816454	1.1748338177	1.2296756453	1.2364014627	1.2941172986	1.2362549195	1.2939639147
0.2	1	1.2429366034	1.2942976200	1.1797086468	1.2284569380	1.2415317592	1.2928347244	1.2413846080	1.2926814926
0.3	1	1.2480727050	1.2930135946	1.1845834759	1.2272382307	1.2466620557	1.2915521503	1.2465142965	1.2913990705
0.4	1	1.2532088067	1.2917295692	1.1894583051	1.2260195235	1.2517923522	1.2902695762	1.2516439849	1.2901166484
0.5	1	1.2583449084	1.2904455438	1.1943331342	1.2248008162	1.2569226488	1.2889870020	1.2567736734	1.2888342263
0.6	1	1.2634810100	1.2891615184	1.1992079633	1.2235821089	1.2620529453	1.2877044279	1.2619033618	1.2875518042
0.7	1	1.2686171117	1.2878774929	1.2040827924	1.2223634016	1.2671832418	1.2864218538	1.2670330503	1.2862693820
0.8	1	1.2737532134	1.2865934675	1.2089576215	1.2211446943	1.2723135383	1.2851392797	1.2721627388	1.2849869599
0.9	1	1.2788893150	1.2853094421	1.2138324507	1.2199259871	1.2774438349	1.2838567055	1.2772924272	1.2837045378
1	1	1.2840254167	1.2840254167	1.2187072798	1.2187072798	1.2825741314	1.2825741314	1.2824221157	1.2824221157

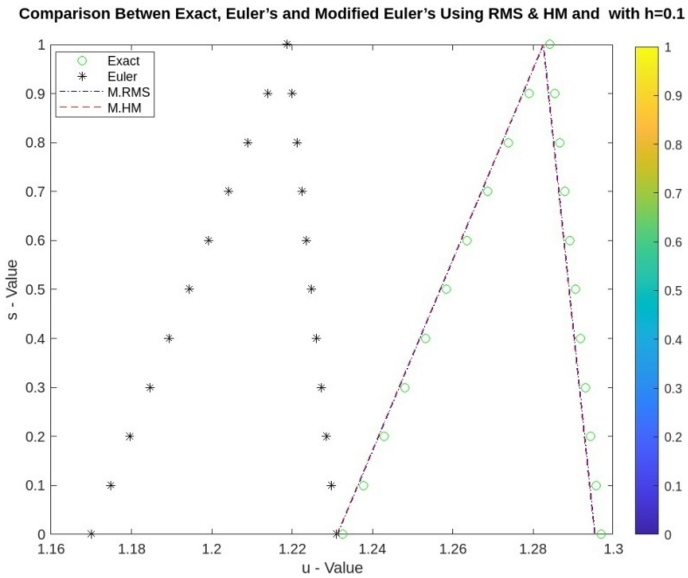

Figure 3.1: Graphical Representation of Solutions for Modified Euler's Method using Root Mean Square and Harmonic Mean with $h = 0.1$

Table 3.2: Error Analysis for Modified Euler's Method using Root Mean Square and Harmonic Mean with $h = 0.1$

γ	Euler's Method		Modified Euler's Method using Root Mean Square		Modified Euler's Method usingHarmonic Mean	
	LOWER	UPPER	Lower	Upper	Lower	Upper
0	0.0627054114	0.0659713183	0.0013932339	0.0014657982	0.0015391689	0.0016193340
0.1	0.0629666840	0.0659060001	0.0013990390	0.0014643468	0.0015455822	0.0016177307
0.2	0.0632279566	0.0658406820	0.0014048442	0.0014628956	0.0015519954	0.0016161274
0.3	0.0634892291	0.0657753639	0.0014106493	0.0014614443	0.0015584085	0.0016145241
0.4	0.0637505016	0.0657100457	0.0014164545	0.0014599930	0.0015648218	0.0016129208
0.5	0.0640117742	0.0656447276	0.0014222596	0.0014585418	0.0015712350	0.0016113175
0.6	0.0642730467	0.0655794095	0.0014280647	0.0014570905	0.0015776482	0.0016097142
0.7	0.0645343193	0.0655140913	0.0014338699	0.0014556391	0.0015840614	0.0016081109
0.8	0.0647955919	0.0654487732	0.0014396751	0.0014541878	0.0015904746	0.0016065076
0.9	0.0650568643	0.0653834550	0.0014454801	0.0014527366	0.0015968878	0.0016049043
1	0.0653181369	0.0653181369	0.0014512853	0.0014512853	0.0016033010	0.0016033010

Table 3.3: Percentage Relative Error Analysis for Modified Euler's Method using Root Mean Square and Harmonic Mean with h = 0.1

γ	Euler's Method	Modified Euler's Method Using Root Mean Square	Modified Euler's Method Using Harmonic Mean
	(Lower, Upper)	(Lower, Upper)	(Lower, Upper)
$\gamma \in [0,1]$	5.09%	0.11%	0.12%

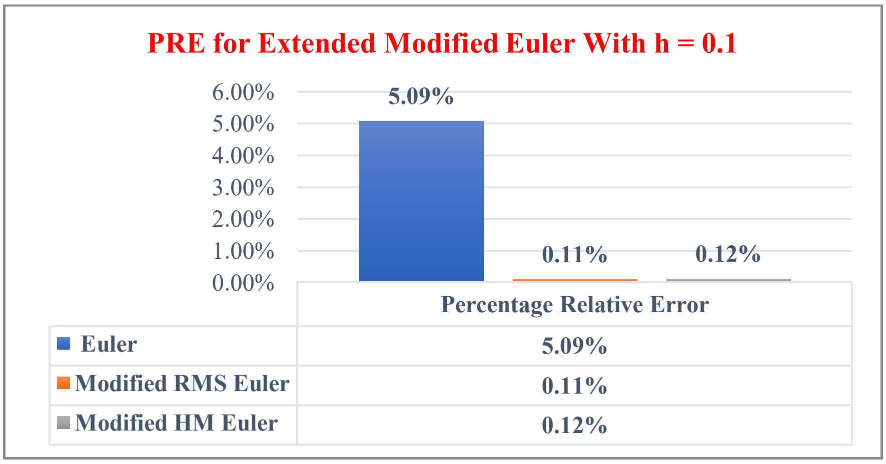

Figure 3.3: Graphical Representation of Percentage Relative Error Analysisfor Modified Euler's Method using Root Mean Squareand Harmonic Mean with h = 0.1

Table 4.1: Numerical Solutions for Modified Euler's Method using Root Mean Square and Harmonic Mean with $h = 0.001$

γ	T	Exact Solutions		Euler's Method		Modified Euler's Method usingRoot Mean Square		Modified Euler's Method using Harmonic Mean	
		LOWER	UPPER	Lower	Upper	Lower	Upper	Lower	Upper
0	1	1.2326644000	1.2968656709	0.9619454992	1.0120468273	0.9619500853	1.0120516522	0.9619500853	1.0120516522
0.1	1	1.2378005017	1.2955816454	0.9659536055	1.0110448008	0.9659582106	1.0110496209	0.9659582106	1.0110496209
0.2	1	1.2429366034	1.2942976200	0.9699617117	1.0100427742	0.9699663360	1.0100475895	0.9699663360	1.0100475895
0.3	1	1.2480727050	1.2930135946	0.9739698180	1.0090407476	0.9739744613	1.0090455582	0.9739744613	1.0090455582
0.4	1	1.2532088067	1.2917295692	0.9779779242	1.0080387211	0.9779825867	1.0080435268	0.9779825867	1.0080435268
0.5	1	1.2583449084	1.2904455438	0.9819860305	1.0070366945	0.9819907120	1.0070414955	0.9819907120	1.0070414955
0.6	1	1.2634810100	1.2891615184	0.9859941367	1.0060346679	0.9859988374	1.0060394642	0.9859988374	1.0060394642
0.7	1	1.2686171117	1.2878774929	0.9900022430	1.0050326414	0.9900069627	1.0050374328	0.9900069627	1.0050374328
0.8	1	1.2737532134	1.2865934675	0.9940103492	1.0040306148	0.9940150881	1.0040354015	0.9940150881	1.0040354015
0.9	1	1.2788893150	1.2853094421	0.9980184554	1.0030285883	0.9980232134	1.0030333701	0.9980232134	1.0030333701
1	1	1.2840254167	1.2840254167	1.0020265617	1.0020265617	1.0020313388	1.0020313388	1.0020313388	1.0020313388

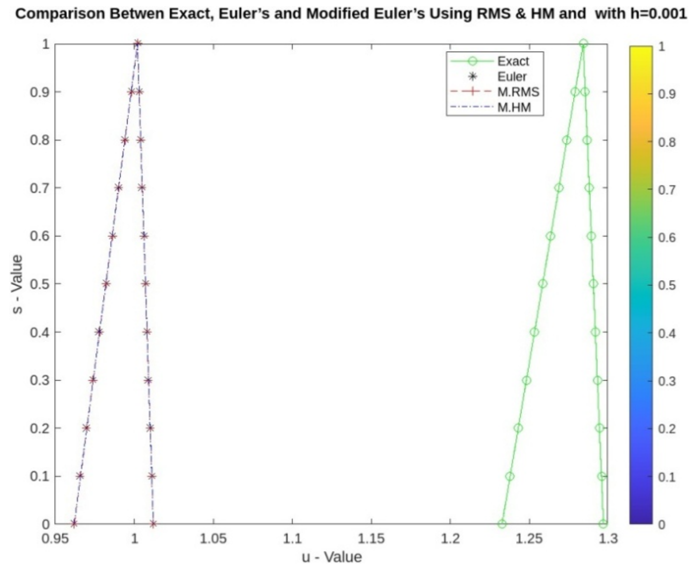

Figure 4.1: Graphical Representation of Solutions for Modified Euler's Method using Root Mean Square and Harmonic Mean with $h = 0.001$

Table 4.2: Error Analysis for Modified Euler's Method using Root Mean Square and Harmonic Mean with $h = 0.001$

γ	Euler's Method		Modified Euler's Method using Root Mean Square		Modified Euler's Method using Harmonic Mean	
	LOWER	UPPER	Lower	Upper	Lower	Upper
0	0.2707189008	0.2848188436	0.2707143147	0.2848140187	0.2707143147	0.2848140187
0.1	0.2718468962	0.2845368446	0.2718422911	0.2845320245	0.2718422911	0.2845320245
0.2	0.2729748917	0.2842548458	0.2729702674	0.2842500305	0.2729702674	0.2842500305
0.3	0.2741028870	0.2839728470	0.2740982437	0.2839680364	0.2740982437	0.2839680364
0.4	0.2752308825	0.2836908481	0.2752262200	0.2836860424	0.2752262200	0.2836860424
0.5	0.2763588779	0.2834088493	0.2763541964	0.2834040483	0.2763541964	0.2834040483
0.6	0.2774868733	0.2831268505	0.2774821726	0.2831220542	0.2774821726	0.2831220542
0.7	0.2786148687	0.2828448515	0.2786101490	0.2828400601	0.2786101490	0.2828400601
0.8	0.2797428642	0.2825628527	0.2797381253	0.2825580660	0.2797381253	0.2825580660
0.9	0.2808708596	0.2822808538	0.2808661016	0.2822760720	0.2808661016	0.2822760720
1	0.2819988550	0.2819988550	0.2819940779	0.2819940779	0.2819940779	0.2819940779

Table 4.3: Percentage Relative Error Analysis for ModifiedEuler's Method using Root Mean Square and Harmonic Mean with h = 0.001

γ	Euler's Method	Modified Euler's Method Using Root Mean Square	Modified Euler's Method Using Harmonic Mean
	(Lower, Upper)	(Lower, Upper)	(Lower, Upper)
$\gamma \in [0, 1]$	**21.96%**	**21.96%**	**21.96%**

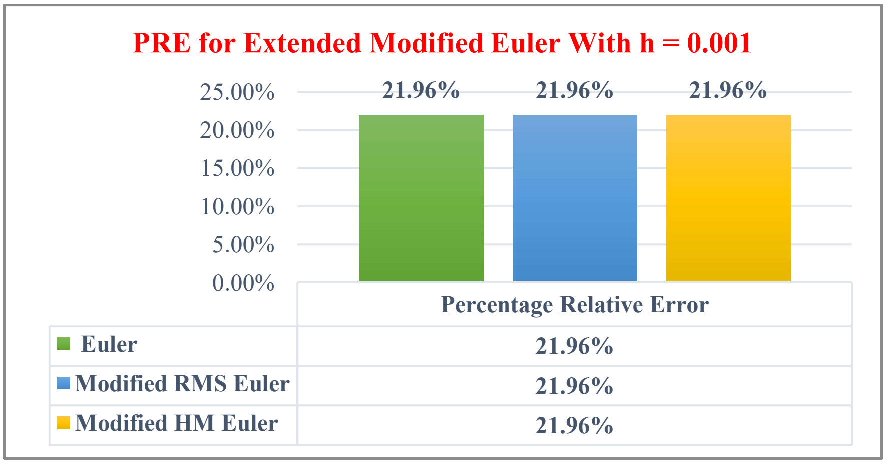

Figure 4.3: Graphical Representation of Percentage Relative Error Analysisfor Modified Euler's Method using Root Mean Squareand Harmonic Mean with h = 0.001

Tables 3.3, 4.3, and figures 3.3, 4.3 indicate that the findings obtained via the Extended Modified Euler Method utilizing the Root Mean Square and Harmonic Mean are closely aligned with the precise value. This methodology may provide more accurate findings than the conventional Euler method. The case study for the two scenarios h = 0.1 and h = 0.001 reveals that the suggested Modified Euler methods minimize error levels well. Consequently, the approaches are superior to the conventional approach.

(A3) Improved Euler's Method in Combination with Root Mean Square and Harmonic Mean

For various values of $\gamma \in [0,1]$, the approximate solutions derived using Root Mean Square and Harmonic Mean of Improved Euler's method for $h = 0.1$ and $h = 0.001$ are shown below:

Table 5.1: Numerical Solutions for Improved Euler's Method using Root Mean Square and Harmonic Mean with $h = 0.1$

γ	T	Exact Solutions		Euler's Method		Improved Euler's Method usingRoot Mean Square		Improved Euler's Method using Harmonic Mean	
		LOWER	UPPER	Lower	Upper	Lower	Upper	Lower	Upper
0	1	1.2326644000	1.2968656709	1.1699589886	1.2308943526	1.2703310936	1.3364941714	1.2664762534	1.3324385582
0.1	1	1.2378005017	1.2955816454	1.1748338177	1.2296756453	1.2756241398	1.3351709099	1.2717532378	1.3311193121
0.2	1	1.2429366034	1.2942976200	1.1797086468	1.2284569380	1.2809171861	1.3338476483	1.2770302222	1.3298000660
0.3	1	1.2480727050	1.2930135946	1.1845834759	1.2272382307	1.2862102323	1.3325243867	1.2823072065	1.3284808199
0.4	1	1.2532088067	1.2917295692	1.1894583051	1.2260195235	1.2915032785	1.3312011252	1.2875841909	1.3271615738
0.5	1	1.2583449084	1.2904455438	1.1943331342	1.2248008162	1.2967963247	1.3298778636	1.2928611753	1.3258423278
0.6	1	1.2634810100	1.2891615184	1.1992079633	1.2235821089	1.3020893710	1.3285546021	1.2981381597	1.3245230817
0.7	1	1.2686171117	1.2878774929	1.2040827924	1.2223634016	1.3073824172	1.3272313405	1.3034151441	1.3232038356
0.8	1	1.2737532134	1.2865934675	1.2089576215	1.2211446943	1.3126754634	1.3259080790	1.3086921285	1.3218845895
0.9	1	1.2788893150	1.2853094421	1.2138324507	1.2199259871	1.3179685096	1.3245848174	1.3139691129	1.3205653434
1	1	1.2840254167	1.2840254167	1.2187072798	1.2187072798	1.3232615559	1.3232615559	1.3192460973	1.3192460973

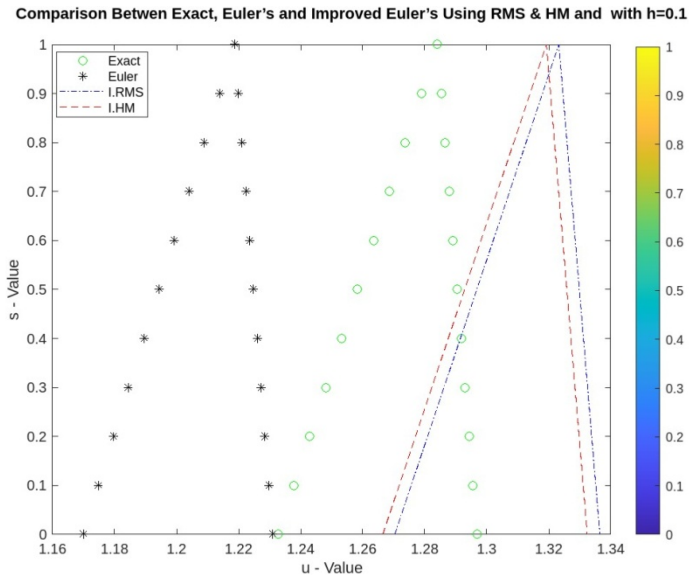

Figure 5.1: Graphical Representation of Solutions for Improved Euler's Method using Root Mean Square and Harmonic Mean with $h = 0.1$

Table 5.2: Error Analysis for Improved Euler's Method using Root Mean Square and Harmonic Mean with $h = 0.1$

γ	Euler's Method		Improved Euler's Method using Root Mean Square		Modified Euler's Method using Harmonic Mean	
	LOWER	**UPPER**	**Lower**	**Upper**	**Lower**	**Upper**
0	0.0627054114	0.0659713183	0.0376666936	0.0396285005	0.0338118534	0.0355728873
0.1	0.0629666840	0.0659060001	0.0378236381	0.0395892645	0.0339527361	0.0355376667
0.2	0.0632279566	0.0658406820	0.0379805827	0.0395500283	0.0340936188	0.0355024460
0.3	0.0634892291	0.0657753639	0.0381375273	0.0395107921	0.0342345015	0.0354672253
0.4	0.0637505016	0.0657100457	0.0382944718	0.0394715560	0.0343753842	0.0354320046
0.5	0.0640117742	0.0656447276	0.0384514163	0.0394323198	0.0345162669	0.0353967840
0.6	0.0642730467	0.0655794095	0.0386083610	0.0393930837	0.0346571497	0.0353615633
0.7	0.0645343193	0.0655140913	0.0387653055	0.0393538476	0.0347980324	0.0353263427
0.8	0.0647955919	0.0654487732	0.0389222500	0.0393146115	0.0349389151	0.0352911220
0.9	0.0650568643	0.0653834550	0.0390791946	0.0392753753	0.0350797979	0.0352559013
1	0.0653181369	0.0653181369	0.0392361392	0.0392361392	0.0352206806	0.0352206806

Table 5.3: Percentage Relative Error Analysis for Improved Euler's Method using Root Mean Square and Harmonic Mean with h = 0.1

γ	**Euler's Method**	**Improved Euler's Method Using Root Mean Square**	**Improved Euler's Method Using Harmonic Mean**
	(Lower, Upper)	**(Lower, Upper)**	**(Lower, Upper)**
$\gamma \in [0, 1]$	5.09%	3.06%	2.74%

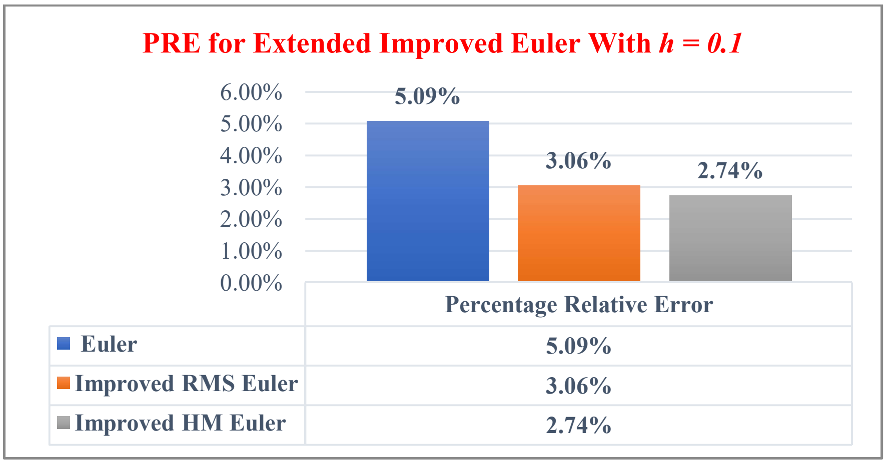

Figure 5.3: Graphical Representation of Percentage Relative Error Analysisfor Improved Euler's Method using Root Mean Square and Harmonic Mean with h = 0.1

Table 6.1: Numerical Solutions for Improved Euler's Method using Root Mean Square and Harmonic Mean with $h = 0.001$

γ	T	Exact Solutions		Euler's Method		Improved Euler's Method usingRoot Mean Square		Improved Euler's Method usingHarmonic Mean	
		LOWER	UPPER	Lower	Upper	Lower	Upper	Lower	Upper
0	1	1.2326644000	1.2968656709	0.9619454992	1.0120468273	0.9619523833	1.0120540699	0.9619523807	1.0120540672
0.1	1	1.2378005017	1.2955816454	0.9659536055	1.0110448008	0.9659605182	1.0110520362	0.9659605157	1.0110520335
0.2	1	1.2429366034	1.2942976200	0.9699617117	1.0100427742	0.9699686532	1.0100500025	0.9699686506	1.0100499998
0.3	1	1.2480727050	1.2930135946	0.9739698180	1.0090407476	0.9739767881	1.0090479687	0.9739767855	1.0090479661
0.4	1	1.2532088067	1.2917295692	0.9779779242	1.0080387211	0.9779849230	1.0080459350	0.9779849204	1.0080459323
0.5	1	1.2583449084	1.2904455438	0.9819860305	1.0070366945	0.9819930579	1.0070439013	0.9819930553	1.0070438986
0.6	1	1.2634810100	1.2891615184	0.9859941367	1.0060346679	0.9860011929	1.0060418675	0.9860011903	1.0060418649
0.7	1	1.2686171117	1.2878774929	0.9900022430	1.0050326414	0.9900093278	1.0050398338	0.9900093252	1.0050398311
0.8	1	1.2737532134	1.2865934675	0.9940103492	1.0040306148	0.9940174627	1.0040378001	0.9940174601	1.0040377974
0.9	1	1.2788893150	1.2853094421	0.9980184554	1.0030285883	0.9980255977	1.0030357663	0.9980255950	1.0030357637
1	1	1.2840254167	1.2840254167	1.0020265617	1.0020265617	1.0020337326	1.0020337326	1.0020337299	1.0020337299

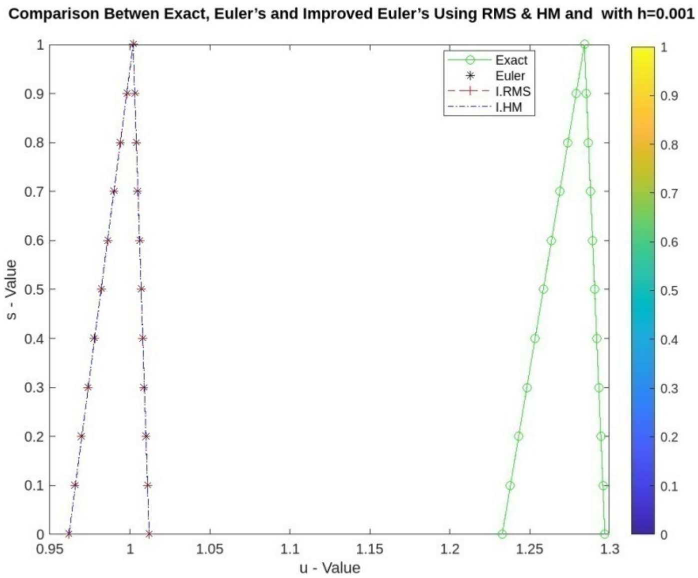

Figure 6.1: Graphical Representation of Solutions for Improved Euler's Method using Root Mean Square and Harmonic Mean with h = 0.001

Table 6.2: Error Analysis for Improved Euler's Method using Root Mean Square and Harmonic Mean with h = 0.001

γ	Euler's Method		Improved Euler's Method using Root Mean Square		Modified Euler's Method using Harmonic Mean	
	LOWER	UPPER	Lower	Upper	Lower	Upper
0	0.2707189008	0.2848188436	0.2707120167	0.2848116010	0.2707120193	0.2848116037
0.1	0.2718468962	0.2845368446	0.2718399835	0.2845296092	0.2718399860	0.2845296119
0.2	0.2729748917	0.2842548458	0.2729679502	0.2842476175	0.2729679528	0.2842476202
0.3	0.2741028870	0.2839728470	0.2740959169	0.2839656259	0.2740959195	0.2839656285
0.4	0.2752308825	0.2836908481	0.2752238837	0.2836836342	0.2752238863	0.2836836369
0.5	0.2763588779	0.2834088493	0.2763518505	0.2834016425	0.2763518531	0.2834016452
0.6	0.2774868733	0.2831268505	0.2774798171	0.2831196509	0.2774798197	0.2831196535
0.7	0.2786148687	0.2828448515	0.2786077839	0.2828376591	0.2786077865	0.2828376618
0.8	0.2797428642	0.2825628527	0.2797357507	0.2825556674	0.2797357533	0.2825556701
0.9	0.2808708596	0.2822808538	0.2808637173	0.2822736758	0.2808637200	0.2822736784
1	0.2819988550	0.2819988550	0.2819916841	0.2819916841	0.2819916868	0.2819916868

Table 6.3: Percentage Relative Error Analysis for Improved Euler's Method using Root Mean Square and Harmonic Mean with h = 0.001

γ	Euler's Method	Improved Euler's Method Using Root Mean Square	Improved Euler's Method Using Harmonic Mean
	(Lower, Upper)	(Lower, Upper)	(Lower, Upper)
$\gamma \in [0,1]$	21.96%	21.96%	21.96%

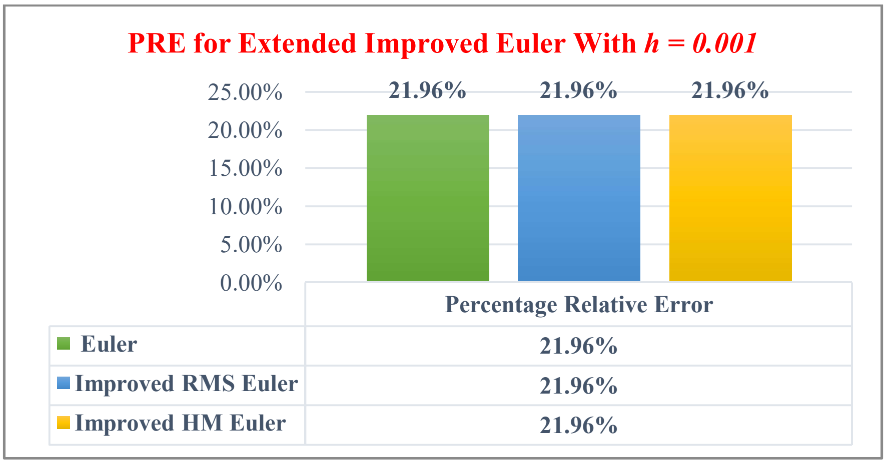

Figure 6.3: Graphical Representation of Percentage Relative Error Analysisfor Improved Euler's Method using Root Mean Square and Harmonic Mean with h = 0.001

Tables 5.3, 6.3, and figures 5.3, 6.3 indicate that the findings obtained via the Extended Improved Euler Method utilizing the Root Mean Square and Harmonic Mean are closely aligned with the precise value. This methodology may provide more accurate findings than the conventional Euler method. The case study for the two scenarios h = 0.1 and h = 0.001 reveals that the suggested Improved Euler methods minimize error levels well. Consequently, the approaches are superior to the conventional approach.

(B) *Error Analysis for Extended Euler's Methods Visualized Graphically Using Root Mean Square and Harmonic Mean with h = 0.1 and h = 0.001:*

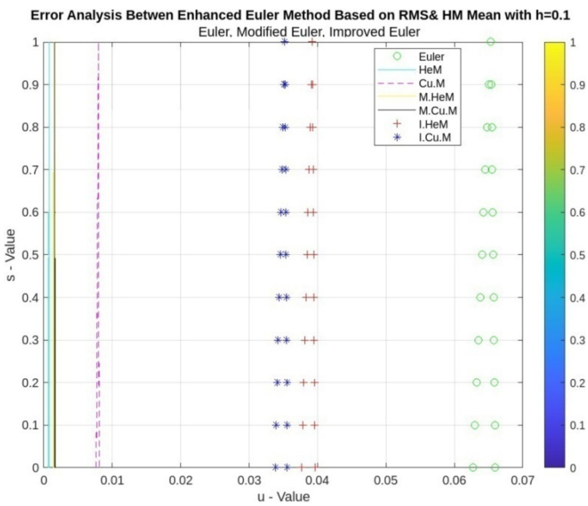

Figure 7: Graphical Representation of Error Analysis for Extended Euler's Method using Root Mean Square and Harmonic Mean with h = 0.1

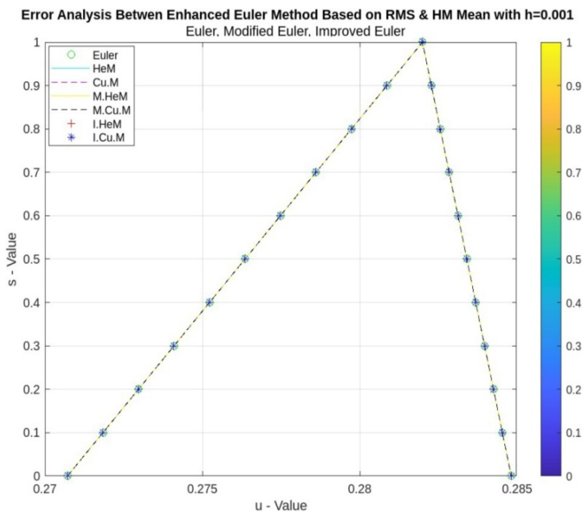

Figure 8: Graphical Representation of Error Analysis for ExtendedEuler's Method using Root Mean Square and Harmonic Mean with h = 0.001

Euler's Method, Modified Euler's Method, and Improved Euler's methods, such as the Harmonic Mean and Root Mean Square, have been proposed to address the Fuzzy initial value problem. The numerical solutions generated by the proposed methods for a variety of γ values within the range of 0 to 1 for h values of 0.1 and 0.001 are illustrated in the tables and figures above. Additionally, the results of Euler's Method are presented to assess the effectiveness of the proposed methods. The investigation determined that, with the exception of Euler's Method, the other methods offer answers that are exceedingly close to the precise values.

6. CONCLUSIONS

This research suggests the use of three important numerical techniques: Euler, Modified Euler, and Improved Euler, to solve fuzzy initial value problems using the Root Mean Square and Harmonic Mean. The comparative analysis of the example's responses demonstrated that the three enhanced methods outperformed the traditional Euler's method. As this study shows, the solutions that were found using the suggested methods are very similar to the exact solutions for $h = 0.1$ and $h = 0.001$. The research presents superior performance in comparison to conventional Euler's approaches.

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